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Relations among supersymmetric lattice gauge theories via orbifolding

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Abstract: We show how to derive Catterall's supersymmetric lattice gauge theories directly from the general principle of orbifolding followed by a variant of the usual deconstruction. These theories are forced to be complexified due to a clash between charge assignments under U(1)-symmetries and lattice assignments in terms of scalar, vector and tensor components for the fermions. Other prescriptions for how to discretize the theory follow automatically by orbifolding and deconstruction. We find that Catterall's complexified model for the two-dimensional $\mathcal{N} = (2, 2)$ theory has two independent preserved supersymmetries. We comment on consistent truncations to lattice theories without this complexification and with the correct continuum limit. The construction of lattice theories this way is general, and can be used to derive new supersymmetric lattice theories through the orbifolding procedure. As an example, we apply the prescription to topologically twisted four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. We show that a consistent truncation is closely related to the lattice formulation previously given by Sugino.

Keywords: Lattice Gauge Field Theories, Lattice Quantum Field Theory, Extended Supersymmetry, Matrix Models.

Contents

1. Introduction

Recently, there has been a rapid series of developments in the lattice construction of supersymmetric gauge theories [1]–[15].[1]

These lattice formulations share one common feature; there is at least one preserved nilpotent scalar supercharge Q, which is a part of the original supersymmetry generators of the continuum theories. Roughly speaking, which supercharge is chosen to be preserved on the lattice determines the lattice formulation. This way of defining supersymmetric lattice gauge theories lies very close to the way topological field theories are defined in the continuum based on BRST symmetry Q. As is well-known, such topological field theories in the continuum can alternatively be viewed as twistings of ordinary field theories with space-time supersymmetry. It is therefore very natural to define lattice gauge theories with some remnant(s) of supersymmetry by means of the same procedure. After an "untwisting" on the lattice one can define physical observables which hopefully will not suffer from the usual fine tuning problems of other approaches to lattice supersymmetry.

In refs. [1]–[4], a systematic way to generate lattice structure from a matrix theory (the "mother theory") has been presented. Here the preserved supercharge is one component of the original supersymmetry generators in general.[2] In this formulation, the space-time lattice itself is generated by orbifolding followed by deconstruction [20], and the dimensionality is determined by the number of the maximal global $\mathrm{U}(1)$ symmetries of the mother

[1]Possible difficulties with these formulations are discussed in [16]–[18].

[2]For a discussion of the relations between these lattice theories and topological field theories, see ref. [19].

theory. Therefore, possible lattice theories generated from a given mother theory are restricted. A classification of orbifolded theories with up to eight supercharges has recently been given in [5].

Among alternative lattice formulations of supersymmetric gauge theories are those due to Catterall [6]–[8] and Sugino [9]–[12], both of which preserve the BRST charge of a topologically twisted supersymmetric gauge theory [21]. The idea of both of these formulations is to write down lattice actions that are Q-exact at fixed lattice spacing. Although they thus seem to be close to each other in spirit, they appear very different in detail at first sight. One surprising feature of Catterall's formulation is that it seems to require a complexification of fields in order to preserve both gauge invariance and some remnant of supersymmetry on the lattice. After constructing the lattice action for the complexified fields, the path-integral has been restricted to the "real line" in actual simulations. By this restriction, however, one breaks both gauge symmetry and the remnant of supersymmetry. Nevertheless, simulations done "on the real line" [23] seem to indicate a surprisingly good approximation to the supersymmetry one hopes to recover in the continuum. Sugino's formulation, on the other hand, does not need this complexification. Yet, both are supposed to be discretizations of the corresponding topological field theories in the continuum. For numerical simulations for Sugino's model, see [13].

Very recently, in a very interesting paper [24], Takimi has shown that the theories of Sugino and the complexified theories of Catterall are indeed connected. More precisely, the degrees of freedom of Catterall's complexified lattice theory for two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory can be reduced in a manner consistent with both gauge symmetry and supersymmetry. The resulting theory is, after some field redefinitions, very closely related to Sugino's lattice formulation.

In this paper, we wish to understand Catterall's theories from the orbifolding procedure. In ref. [5], we derived what we believe is the complete classification of orbifolded theories with up to eight supercharges and none of these theories seemed to include those of Catterall. Is the orbifolding technique not the most general way to generate such supersymmetric lattice theories? Or was the classification incomplete? As we shall show, the answer lies in the restrictions one imposes on oneself if one insists on a particular assignment of fields on the lattice. In particular, the crucial part is the way one insists on identifying fields transforming irreducibly under Lorentz transformations. If one *beforehand* insists on scalars, vectors and tensors in the continuum being represented by site variables, links and corner variables, respectively, then one may run into clashes with the orbifolding technique. This is because the assignment of U(1)-charges (some of which are subsets of Lorentz symmetries) is in a one-to-one correspondence with the generation of the lattice itself. In the case of Catterall's prescription, these U(1)-charges do not match those required for the lattice assignments that are being insisted upon. The apparently only way out is to *complexify*.[3] As we shall show, this can be done so that it introduces just the right amount of additional U(1)-symmetries. The price one pays is that one is not considering the right theory anymore, but a complexified one.

[3]For another way to make connection with the orbifolding procedure, see ref. [24].

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Having understood that this is the way to generate the complexified supersymmetric theories according to Catterall's prescription, it is now a simple matter to generalize this to many other theories. In particular, there is apparently no deeper need to tie oneself up to theories that admit a complete description in terms of Dirac-Kähler fields. If one allows oneself to complexify, many other theories are possible. We shall illustrate this by showing how to generate a complexified version of $\mathcal{N} = 2$ supersymmetric lattice gauge theory in four dimensions by a combination of complexification and orbifolding. As for Catterall's examples, going to the real line breaks both gauge symmetry and the last remnant of lattice supersymmetry.[4] Instead, we demonstrate that we can truncate to fewer degrees of freedom while preserving both gauge symmetry and supersymmetry, just as was done in ref. [24]. The obtained theory is again essentially, up to a few additional terms, equal to Sugino's formulation of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [9].

Our paper is organized as follows. In section 2 we show how to derive Catterall's complexified (2,2)-model by combining orbifolding and complexification. Surprisingly, we find that this theory actually is invariant under two different scalar supercharges $Q_\pm$, not just one as previously believed. The two charges Q_+ and Q_- can be viewed as BRST and anti-BRST charges, respectively, and the action is exact in both of them. We discuss the problems that arise if one tries to project the resulting complexified theory onto the real line: loss of lattice supersymmetry in both the action and the measure (and the combination of the two). In section 3 we comment on the recent observation by Takimi [24] of a consistent truncation of Catterall's complexified model that turns out to be closely related to Sugino's [10]. Because of the existence of two independently conserved supersymmetry charges, we can consider the same type of truncation based on the other supersymmetry charge. As it turns out, it yields the same action, up to trivial changes of conventions. In section 4 we discuss possible generalizations of Catterall's complexified models that can be constructed by orbifolding. This includes many supersymmetric theories that could not be derived by orbifolding in the conventional way, including $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in four dimensions. The challenge is then to find either consistent truncations, or truncations that, although they may break all supersymmetries, may still yield supersymmetric field theories in the continuum without the need of fine tuning. We show that we obtain a theory very closely related to Sugino's formulation of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [9] by one particular truncation, followed by field redefinitions. In section 5 we present our conclusions.

2. Catterall's construction from orbifolding

In this section we show how to obtain Catterall's complexified lattice gauge theories by the orbifolding procedure of refs. [1]–[5]. In particular, we show that the discretization prescription given in [7] can be clearly understood by this procedure. To be definite, we concentrate on the lattice theory for two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry in the

[4] But if the numerical experience of ref. [23] holds here too, this may be a quite good approximation to such supersymmetric lattice gauge theories.

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continuum limit. As part of our derivation, we will also show that there is an additional, hidden, (anti-)BRST-like symmetry in Catterall's model.

2.1 Derivation of Catterall's action by the orbifolding procedure

As usual with orbifolding technique, we begin with a "mother theory", here a matrix model obtained by dimensional reduction of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in four-dimensional Euclidean space-time,

$$S=\frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[v_\alpha,v_\beta]^2+\frac{i}{2}\bar{\Psi}\Gamma_\alpha[v_\alpha,\Psi]\right),\qquad(\alpha,\beta=0,\ldots,3)\tag{2.1}$$

where Γ_α are SO(4) Dirac matrices, v_α are $kN^2\times kN^2$ hermitian matrices, Ψ is a four-component fermion and $\bar{\Psi}\equiv\Psi^TC$ with the charge conjugation matrix C satisfying $C^{-1}\Gamma_\alpha C=-\Gamma_\alpha^T$. Following [2], we choose the notation of the γ-matrices and the charge conjugation matrix as

$$\Gamma_\alpha=\begin{pmatrix}0&\sigma_\alpha\\ \bar{\sigma}_\alpha&0\end{pmatrix},\qquad C=\begin{pmatrix}i\tau_2&0\\ 0&-i\tau_2\end{pmatrix},\tag{2.2}$$

with $\sigma_\alpha=(\mathbf{1},-i\tau_i)$ and $\bar{\sigma}_\alpha=(\mathbf{1},i\tau_i)$ where τ_i $(i=1,2,3)$ are Pauli matrices. Our purpose in this section is to obtain a lattice regularization of topologically twisted two-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauge theory. To this end, we rearrange the fields so that the symmetry of the two-dimensional theory becomes manifest:

$$\begin{gathered}v_0\equiv A_1,\quad v_3\equiv-A_2,\quad v_1+iv_2\equiv i\phi,\quad v_1-iv_2\equiv-i\overline{\phi},\\ \Psi^{(1)}\equiv\begin{pmatrix}-i\chi_{12}-\frac{1}{2}\eta\\ \psi_1-i\psi_2\end{pmatrix},\quad\Psi^{(2)}\equiv\begin{pmatrix}-i\chi_{12}+\frac{1}{2}\eta\\ \psi_1+i\psi_2\end{pmatrix},\end{gathered}\tag{2.3}$$

where we have set $\Psi^T\equiv\left(\Psi^{(1)T},\Psi^{(2)T}\right)$. Then the action (2.2) can be rewritten as

$$\begin{aligned}S=\frac{1}{g^2}\mathrm{Tr}\Big\{&-B_{\mu\nu}^2+iB_{\mu\nu}[A_\mu,A_\nu]-\frac{1}{2}[A_\mu,\phi][A_\mu,\overline{\phi}]+\frac{1}{8}[\phi,\overline{\phi}]^2-i\eta[A_\mu,\psi_\mu]\\ &-i\chi_{\mu\nu}\left([A_\mu,\psi_\nu]-[A_\nu,\psi_\mu]\right)-\frac{i}{4}\eta[\phi,\eta]+i\psi_\mu\left[\overline{\phi},\psi_\mu\right]-\frac{i}{2}\chi_{\mu\nu}\left[\phi,\chi_{\mu\nu}\right]\Big\}\end{aligned}\tag{2.4}$$

where $\chi_{12}=-\chi_{21}$ and we have introduced an auxiliary field $B_{\mu\nu}=-B_{\nu\mu}$. As discussed in [7], we should regard ϕ and $\overline{\phi}$ as independent hermitian matrices rather than complex conjugate. In the expression (2.4), a scalar supersymmetry (equivalently, a BRST symmetry) is manifest, and we can rewrite the action in a Q-exact form as

$$S=\frac{1}{g^2}\mathrm{Tr}\,Q\Big\{-\chi_{\mu\nu}\left(B_{\mu\nu}-i[A_\mu,A_\nu]\right)+i\psi_\mu[A_\mu,\overline{\phi}]+\frac{i}{4}\eta[\phi,\overline{\phi}]\Big\},\tag{2.5}$$

where $B_{\mu\nu}$ is a auxiliary field and Q is the BRST charge which acts on the fields as

$$\begin{aligned}QA_\mu&=\psi_\mu, & Q\psi_\mu&=\frac{i}{2}[A_\mu,\phi],\\ Q\overline{\phi}&=\eta, & Q\eta&=-\frac{i}{2}[\phi,\overline{\phi}],\\ Q\chi_{\mu\nu}&=B_{\mu\nu}, & QB_{\mu\nu}w&=-\frac{i}{2}[\phi,\chi_{\mu\nu}],\qquad Q\phi=0.\end{aligned}\tag{2.6}$$

One can easily show that $Q^2 = \delta_{-\phi/2}$, where δ_θ is the gauge transformation with a parameter θ. Thus, Q is nilpotent up to gauge transformations.

Next, we would like to derive a lattice theory from the mother theory (2.5) using orbifolding and deconstruction while preserving the BRST charge Q. To do so, we must first specify two U(1) symmetries to create a two-dimensional lattice. (For details, see [1]–[5].) In our case, we must demand of these U(1) symmetries that the BRST operator Q has zero charges and all fields have definite charges so that the action (2.5) has zero charge. However, we immediately see that it is impossible. In fact, since the gauge fields A_μ should become link variables, they must have non-zero charges. Then, from the BRST transformation (2.6), ψ_μ must have the same U(1) charges as A_μ, while $\{\phi, \overline{\phi}, \eta\}$ should have zero charges. Under this condition, the U(1) charges of the second term of (2.7) cannot be zero; it is impossible to assign non-vanishing definite U(1) charges to the fields. This is consistent with our earlier result [5] that the two-dimensional lattice theory constructed by orbifolding from the mother theory (2.1) is unique, and coincides with the one given in [2].

In order to avoid this problem, we extend, as is done in ref. [7], all fields except ϕ and $\overline{\phi}$ to complex matrices, and we change simultaneously the action (2.5) as follows:

$$
\begin{aligned}
S =& \frac{1}{2g^2} \mathrm{Tr}\, Q_+ \Big\{ \chi^\dagger_{\mu\nu}\Big(-B_{\mu\nu} + i[A_\mu, A_\nu]\Big) + \chi_{\mu\nu}\Big(-B^\dagger_{\mu\nu} + i[A^\dagger_\mu, A^\dagger_\nu]\Big) \\
& \qquad + i\psi^\dagger_\mu[A_\mu, \overline{\phi}] + i\psi_\mu[A^\dagger_\mu, \overline{\phi}] + \frac{i}{4}\eta_+[\phi, \overline{\phi}] + \frac{1}{2}\eta_- d \Big\},
\end{aligned} \tag{2.7}
$$

where $A^\dagger_\mu$, $B^\dagger_{\mu\nu}$ and $\psi^\dagger_\mu$ are hermitian conjugate of A_μ, $B_{\mu\nu}$ and ψ_μ, respectively, η_+ and η_- are independent hermitian matrices and d is a hermitian auxiliary field. The BRST charge Q_+ is a natural extension of Q in (2.6) which act to the fields as

$$
\begin{aligned}
Q_+ A_\mu &= \psi_\mu, & Q_+\psi_\mu &= \frac{i}{2}[A_\mu, \phi], \\
Q_+ A^\dagger_\mu &= \psi^\dagger_\mu, & Q_+\psi^\dagger_\mu &= \frac{i}{2}[A^\dagger_\mu, \phi], \\
Q_+\overline{\phi} &= \eta_+, & Q_+\eta_+ &= -\frac{i}{2}[\phi, \overline{\phi}], \\
Q_+ d &= -\frac{i}{2}[\phi, \eta_-], & Q_+\eta_- &= d, \\
Q_+\chi_{\mu\nu} &= B_{\mu\nu}, & Q_+ B_{\mu\nu} &= -\frac{i}{2}[\phi, \chi_{\mu\nu}], \\
Q_+\chi^\dagger_{\mu\nu} &= B^\dagger_{\mu\nu}, & Q_+ B^\dagger_{\mu\nu} &= -\frac{i}{2}[\phi, \chi^\dagger_{\mu\nu}], \qquad Q_+\phi = 0.
\end{aligned} \tag{2.8}
$$

The charge Q_+ is nilpotent up to gauge transformations, just as was the original Q. It is easy to see that (2.7) returns to the original form (2.4) if we take $A^\dagger_\mu = A_\mu$, $B^\dagger_{\mu\nu} = B_{\mu\nu}$, $\psi^\dagger_\mu = \psi_\mu$, $d = \eta_- = 0$ and $\eta_+ = \eta$.

By the above extension, the action acquires extra U(1) symmetries and the action is invariant under the transformation,

$$
\Phi \to e^{iq_1\theta_1 + iq_2\theta_2}\Phi, \qquad (\theta_1, \theta_2 \in [0, 2\pi)) \tag{2.9}
$$

	A_1	A_2	ϕ	$\overline{\phi}$	B_{12}	η_+	η_-	ψ_1	ψ_2	χ_{12}
q_1	1	0	0	0	1	0	0	1	0	1
q_2	0	1	0	0	1	0	0	0	1	1

Table 1: The charge assignment for the complexified fields

where Φ is a collective field content in the action (2.7), and the U(1) charges q_1 and q_2 are given in table 1. For the purpose of the future discussion, we introduce two vectors,

$$\mathbf{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.10}$$

As discussed in [5], the orbifolded action is obtained by substituting the following expansion of the fields in (2.7):

$$\begin{aligned}
A_\mu &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} A_\mu(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}+\mathbf{e}_\mu}, & A_\mu^\dagger &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} A_\mu^\dagger(\mathbf{n}) \otimes E_{\mathbf{n}+\mathbf{e}_\mu,\mathbf{n}}, \\
\phi &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \phi(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}}, & \overline{\phi} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \overline{\phi}(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}}, \\
B_{12} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} B_{12}(\mathbf{n}) \otimes E_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2,\mathbf{n}}, & B_{12}^\dagger &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} B_{12}^\dagger(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}, \\
\eta_+ &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \eta_+(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}}, & \eta_- &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \eta_-(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}}, \\
\psi_\mu &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \psi_\mu(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}+\mathbf{e}_\mu}, & \psi_\mu^\dagger &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \psi_\mu^\dagger(\mathbf{n}) \otimes E_{\mathbf{n}+\mathbf{e}_\mu,\mathbf{n}}, \\
\chi_{12} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \chi_{12}(\mathbf{n}) \otimes E_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2,\mathbf{n}}, & \chi_{12}^\dagger &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \chi_{12}^\dagger(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}, \\
d &= \sum_{\mathbf{n}\in\mathbb{Z}_N^2} d(\mathbf{n}) \otimes E_{\mathbf{n},\mathbf{n}},
\end{aligned} \tag{2.11}$$

where $E_{\mathbf{m},\mathbf{n}}$ ($\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2)$) is an $N^2 \times N^2$ matrix defined by

$$E_{\mathbf{m},\mathbf{n}} \equiv E_{m_1,n_1} \otimes E_{m_2,n_2}. \quad \Big((E_{i,j})_{kl} = \delta_{ik}\delta_{jl}, \quad i,j,k,l = 1,\ldots,N \Big) \tag{2.12}$$

Furthermore, in the standard method of deconstruction, we search for flat directions, and use these to shift appropriate combinations of fields in order to generate kinetic terms. Here we wish to shift the fields A_μ and $A_\mu^\dagger$ with the amount of $1/a$ in order to introduce such kinetic terms for the gauge potentials, and by gauge symmetry, all other fields with non-trivial couplings to these gauge potentials. Instead of this shift operation, however, we could replace $A_\mu(\mathbf{n})$ and $A_\mu^\dagger(\mathbf{n})$ as [19]

$$\begin{aligned}
A_\mu(\mathbf{n}) &\to \frac{1}{ia} e^{iaA_\mu(\mathbf{n})} \equiv -iU_\mu(\mathbf{n}), \\
A_\mu^\dagger(\mathbf{n}) &\to -\frac{1}{ia} e^{-iaA_\mu^\dagger(\mathbf{n})} \equiv iU_\mu^\dagger(\mathbf{n}).
\end{aligned} \tag{2.13}$$

To leading order in the dimensionful quantity a, this is equivalent up to the usual shift prescription. In particular, in the naive continuum limit we cannot tell the difference. Note, however, that $U_\mu(\mathbf{n})$ and $U_\mu^\dagger(\mathbf{n})$ are not unitary matrices since $A_\mu(\mathbf{n})$ and $A_\mu^\dagger(\mathbf{n})$ are not hermitian. This point is crucial for what follows. For the moment, we can choose to view the change $A_\mu(\mathbf{n}) \to U_\mu(\mathbf{n})$ as simply a change of notation, since both $A_\mu(\mathbf{n})$ and $U_\mu(\mathbf{n})$ (although it notation-wise resembles a unitary link) are integrated over as complex matrices.

As a result of these manipulations, we obtain a lattice action,

$$\begin{aligned} S = \frac{1}{2g^2}\mathrm{Tr}\, Q_+ \sum_{\mathbf{n}\in\mathbb{Z}_N^2} \Big\{ & \chi_{\mu\nu}^\dagger(\mathbf{n})\Big[-B_{\mu\nu}(\mathbf{n}) - i\Big(U_\mu(\mathbf{n})U_\nu(\mathbf{n}+\mathbf{e}_\mu) - U_\nu(\mathbf{n})U_\mu(\mathbf{n}+\mathbf{e}_\nu)\Big)\Big] \\ & + \chi_{\mu\nu}(\mathbf{n})\Big[-B_{\mu\nu}^\dagger(\mathbf{n}) - i\Big(U_\mu^\dagger(\mathbf{n}+\mathbf{e}_\nu)U_\nu^\dagger(\mathbf{n}) - U_\nu^\dagger(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n})\Big)\Big] \\ & - \psi_\mu^\dagger(\mathbf{n})\Big(U_\mu(\mathbf{n})\overline{\phi}(\mathbf{n}+\mathbf{e}_\mu) - \overline{\phi}(\mathbf{n})U_\mu(\mathbf{n})\Big) \\ & - \psi_\mu(\mathbf{n})\Big(U_\mu^\dagger(\mathbf{n})\overline{\phi}(\mathbf{n}) - \overline{\phi}(\mathbf{n}+\mathbf{e}_\mu)U_\mu(\mathbf{n})\Big) \\ & + \frac{i}{4}\eta_+(\mathbf{n})[\phi(\mathbf{n}),\overline{\phi}(\mathbf{n})] + \frac{1}{2}\eta_-(\mathbf{n})d(\mathbf{n})\Big\}, \end{aligned} \tag{2.14}$$

where the BRST transformation (2.8) becomes as

$$\begin{aligned} Q_+U_\mu(\mathbf{n}) &= i\psi_\mu(\mathbf{n}), & Q_+\psi_\mu(\mathbf{n}) &= \frac{1}{2}\Big(U_\mu(\mathbf{n})\phi(\mathbf{n}+\mathbf{e}_\mu) - \phi(\mathbf{n})U_\mu(\mathbf{n})\Big), \\ Q_+U_\mu^\dagger(\mathbf{n}) &= -i\psi_\mu^\dagger(\mathbf{n}), & Q_+\psi_\mu^\dagger(\mathbf{n}) &= -\frac{1}{2}\Big(U_\mu^\dagger(\mathbf{n})\phi(\mathbf{n}) - \phi(\mathbf{n}+\mathbf{e}_\mu)U_\mu(\mathbf{n})\Big), \\ Q_+\overline{\phi}(\mathbf{n}) &= \eta_+(\mathbf{n}), & Q_+\eta_+(\mathbf{n}) &= -\frac{i}{2}[\phi(\mathbf{n}),\overline{\phi}(\mathbf{n})], \\ Q_+d(\mathbf{n}) &= -\frac{1}{2}[\phi(\mathbf{n}),\eta_-(\mathbf{n})], & Q_+\eta_-(\mathbf{n}) &= d(\mathbf{n}), \\ Q_+\chi_{\mu\nu}(\mathbf{n}) &= B_{\mu\nu}(\mathbf{n}), & Q_+B_{\mu\nu} &= -\frac{i}{2}\Big(\phi(\mathbf{n})\chi_{\mu\nu}(\mathbf{n}) - \chi_{\mu\nu}(\mathbf{n})\phi(\mathbf{n}+\mathbf{e}_\mu+\mathbf{e}_\nu)\Big), \\ Q_+\chi_{\mu\nu}^\dagger(\mathbf{n}) &= B_{\mu\nu}^\dagger(\mathbf{n}), & Q_+B_{\mu\nu}^\dagger(\mathbf{n}) &= -\frac{i}{2}\Big(\phi(\mathbf{n}+\mathbf{e}_\mu+\mathbf{e}_\nu)\chi_{\mu\nu}^\dagger(\mathbf{n}) - \chi_{\mu\nu}^\dagger(\mathbf{n})\phi(\mathbf{n})\Big), \\ Q_+\phi(\mathbf{n}) &= 0. \end{aligned} \tag{2.15}$$

Integrating out the auxiliary field $d(\mathbf{n})$, the action (2.14) is nothing but that of the lattice gauge theory given in [7]. We emphasize that the prescription given in [7] is automatically reproduced by a combination of orbifolding and the variant of deconstruction described above.

2.2 Enhancement of symmetry by complexification

The complexification of both bosonic and fermionic fields is reminiscent of a balanced doubling of degrees of freedom on both the bosonic and fermionic sides, and one is tempted to search for a corresponding enhancement of supersymmetry. Indeed, we can show that the complexified action (2.7) possesses another BRST-like symmetry, similar to the often encountered additional anti-BRST symmetries of topological theories in the continuum. In fact, the action can be rewritten as

$$S = \frac{1}{2g^2}\mathrm{Tr}\left\{Q_+Q_-\Big(\frac{1}{2}\eta_-\eta_+ + 2\psi_\mu^\dagger\psi_\mu - \chi_{\mu\nu}^\dagger\chi_{\mu\nu}\Big) + Q_+\Big(i\chi_{\mu\nu}^\dagger\,[A_\mu, A_\nu] + i\chi_{\mu\nu}[A_\mu^\dagger, A_\nu^\dagger]\Big)\right\}, \tag{2.16}$$

where Q_- acts on the fields as

$$\begin{aligned}
Q_-A_\mu &= \psi_\mu, & Q_-\psi_\mu &= -\frac{i}{2}[A_\mu, \phi], \\
Q_-A_\mu^\dagger &= -\psi_\mu^\dagger, & Q_-\psi_\mu^\dagger &= \frac{i}{2}[A_\mu^\dagger, \phi], \\
Q_-\overline{\phi} &= \eta_-, & Q_-\eta_+ &= -d, \\
Q_-d &= -\frac{i}{2}[\phi, \eta_+], & Q_-\eta_- &= \frac{i}{2}[\phi, \overline{\phi}], \\
Q_-\chi_{\mu\nu} &= -B_{\mu\nu}, & Q_-B_{\mu\nu} &= -\frac{i}{2}[\phi, \chi_{\mu\nu}], \\
Q_-\chi_{\mu\nu}^\dagger &= B_{\mu\nu}^\dagger, & Q_-B_{\mu\nu}^\dagger &= \frac{i}{2}[\phi, \chi_{\mu\nu}^\dagger], \qquad Q_-\phi = 0,
\end{aligned} \tag{2.17}$$

and one can show that the second term of (2.16) is also Q_--closed, *i.e.* it is also manifestly Q_--invariant. In fact, the second term can be expressed as

$$Q_-\Big(i\chi_{\mu\nu}^\dagger[A_\mu, A_\nu] - i\chi_{\mu\nu}[A_\mu^\dagger, A_\nu^\dagger]\Big). \tag{2.18}$$

Q_- is also nilpotent up to gauge transformations and the two operators satisfy

$$\{Q_+, Q_-\} = 0\,, \tag{2.19}$$

just like BRST and anti-BRST charges.

Note that the two supercharges Q_+ and Q_- are actually independent of each other, although the transformations (2.8) and (2.17) look quite similar. One way to see this is to use the relation between Catterall's complex model and the orbifolded theory for two-dimensional $\mathcal{N} = (4, 4)$ supersymmetric gauge theory [24]. In the original orbifolded theory, there are two independent supercharges Q and $\bar{Q}$, and they are not broken by the truncation made in [24]. Using them, $Q_\pm$ can be written as $Q_\pm = (Q \pm \bar{Q})/2$.

Correspondingly, the lattice action (2.14) can be compactly written as

$$\begin{aligned}
S = \frac{1}{2g^2}\mathrm{Tr}\sum_{\mathbf{n}\in\mathbb{Z}_N^2}\Big\{&Q_+Q_-\Big(\frac{1}{2}\eta_-(\mathbf{n})\eta_+(\mathbf{n}) + 2\psi_\mu^\dagger(\mathbf{n})\psi_\mu(\mathbf{n}) - \chi_{\mu\nu}^\dagger(\mathbf{n})\chi_{\mu\nu}(\mathbf{n})\Big) \\
&+Q_+\Big(\chi_{\mu\nu}^\dagger(\mathbf{n})\Big[-B_{\mu\nu}(\mathbf{n}) - i\Big(U_\mu(\mathbf{n})U_\nu(\mathbf{n}+\mathbf{e}_\mu) - U_\nu(\mathbf{n})U_\mu(\mathbf{n}+\mathbf{e}_\nu)\Big)\Big] \\
&+\chi_{\mu\nu}(\mathbf{n})\Big[-B_{\mu\nu}^\dagger(\mathbf{n}) - i\Big(U_\mu^\dagger(\mathbf{n}+\mathbf{e}_\nu)U_\nu^\dagger(\mathbf{n}) - U_\nu^\dagger(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n})\Big)\Big]\Big)\Big\},
\end{aligned} \tag{2.20}$$

where the BRST charge Q_- acts in the following manner:

$$\begin{aligned}
Q_-U_\mu(\mathbf{n}) &= i\psi_\mu(\mathbf{n}), & Q_-\psi_\mu(\mathbf{n}) &= -\frac{1}{2}\Big(U_\mu(\mathbf{n})\phi(\mathbf{n}+\mathbf{e}_\mu) - \phi(\mathbf{n})U_\mu(\mathbf{n})\Big),\\
Q_-U_\mu^\dagger(\mathbf{n}) &= i\psi_\mu^\dagger(\mathbf{n}), & Q_-\psi_\mu^\dagger(\mathbf{n}) &= -\frac{1}{2}\Big(U_\mu^\dagger(\mathbf{n})\phi(\mathbf{n}) - \phi(\mathbf{n}+\mathbf{e}_\mu)U_\mu(\mathbf{n})\Big),\\
Q_-\overline{\phi}(\mathbf{n}) &= \eta_-(\mathbf{n}), & Q_-\eta_+(\mathbf{n}) &= -d(\mathbf{n}),\\
Q_-d(\mathbf{n}) &= -\frac{1}{2}[\phi(\mathbf{n}),\eta_+(\mathbf{n})], & Q_-\eta_-(\mathbf{n}) &= \frac{i}{2}[\phi(\mathbf{n}),\overline{\phi}(\mathbf{n})],\\
Q_-\chi_{\mu\nu}(\mathbf{n}) &= -B_{\mu\nu}(\mathbf{n}), & Q_-B_{\mu\nu}(\mathbf{n}) &= -\frac{i}{2}\Big(\phi(\mathbf{n})\chi_{\mu\nu}(\mathbf{n}) - \chi_{\mu\nu}(\mathbf{n})\phi(\mathbf{n}+\mathbf{e}_\mu+\mathbf{e}_\nu)\Big),\\
Q_-\chi_{\mu\nu}^\dagger(\mathbf{n}) &= B_{\mu\nu}^\dagger(\mathbf{n}), & Q_-B_{\mu\nu}^\dagger(\mathbf{n}) &= \frac{i}{2}\Big(\phi(\mathbf{n}+\mathbf{e}_\mu+\mathbf{e}_\nu)\chi_{\mu\nu}^\dagger(\mathbf{n}) - \chi_{\mu\nu}^\dagger(\mathbf{n})\phi(\mathbf{n})\Big),\\
Q_-\phi(\mathbf{n}) &= 0.
\end{aligned} \tag{2.21}$$

2.3 Naive reduction back to the real line

Because complexification played such a crucial role in deriving the supersymmetric lattice action (2.14), we should expect difficulties if we *a posteriori* reduce fields from the complex plane back to the real line. Indeed, there are problems at many different levels. Let us first consider the lattice gauge symmetry of the complexified action. From the orbifolding procedure the ultralocal $\mathrm{U}(k)$ symmetry of the zero-dimensional mother theory becomes a lattice gauge symmetry, where fields transform as either adjoints or bifundamentals, *viz.*,

$$\begin{aligned}
U_\mu(\mathbf{n}) &\to V^\dagger(\mathbf{n})U_\mu(\mathbf{n})V(\mathbf{n}+\mathbf{e}_\mu), & U_\mu^\dagger(\mathbf{n}) &\to V^\dagger(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n})V(\mathbf{n}),\\
\phi(\mathbf{n}) &\to V^\dagger(\mathbf{n})\phi(\mathbf{n})V(\mathbf{n}), & \overline{\phi}(\mathbf{n}) &\to V^\dagger(\mathbf{n})\overline{\phi}(\mathbf{n})V(\mathbf{n}),\\
B_{12}(\mathbf{n}) &\to V^\dagger(\mathbf{n})B_{12}(\mathbf{n})V(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2), & B_{12}^\dagger(\mathbf{n}) &\to V^\dagger(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)B_{12}^\dagger(\mathbf{n})V(\mathbf{n}),\\
\psi_\mu(\mathbf{n}) &\to V^\dagger(\mathbf{n})\psi_\mu(\mathbf{n})V(\mathbf{n}+\mathbf{e}\mu), & \psi_\mu^\dagger(\mathbf{n}) &\to V^\dagger(\mathbf{n}+\mathbf{e}_\mu)\psi_\mu^\dagger(\mathbf{n})V(\mathbf{n}),\\
\eta_\pm(\mathbf{n}) &\to V^\dagger(\mathbf{n})\eta_\pm(\mathbf{n})V(\mathbf{n}), & d(\mathbf{n}) &\to V^\dagger(\mathbf{n})d(\mathbf{n})V(\mathbf{n}),\\
\chi_{12}(\mathbf{n}) &\to V^\dagger(\mathbf{n})\chi_{12}(\mathbf{n})V(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2), & \chi_{12}^\dagger(\mathbf{n}) &\to V^\dagger(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)\chi_{12}^\dagger(\mathbf{n})V(\mathbf{n}),
\end{aligned} \tag{2.22}$$

where $V \in \mathrm{U}(k)$. In a first attempt at projecting onto the real axis, one could consider [7] taking $A_\mu(\mathbf{n})$ hermitian, and hence $U_\mu(\mathbf{n})$ unitary. This does not alter the gauge transformation for U_μ. But reducing the other fields from being complex to being hermitian is not compatible with the $\mathrm{U}(k)$ symmetry. For instance, requiring $\psi_\mu(\mathbf{n}) = \psi_\mu^\dagger(\mathbf{n})$ is clearly incompatible with the general gauge transformation rule (2.22).

Another difficulty with a naive reduction to the real line is the breaking of the BRST-anti-BRST symmetries. Clearly, if we take $A_\mu(\mathbf{n})$ to be hermitian, and thus $U_\mu(\mathbf{n})$ unitary, the supersymmetry transformations $Q_\pm U_\mu(\mathbf{n}) = i\psi_\mu(\mathbf{n})$ and $Q_\pm U_\mu^\dagger(\mathbf{n}) = \mp i\psi_\mu^\dagger(\mathbf{n})$ are incompatible with the unitarity constraint $U_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}) = 1$. One consequence of this incompatibility is a breaking of the remnants of supersymmetry already at the action level. This is as expected, since one must impose the unitarity constraint $U_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}) = 1$ in the action, while one needs $Q_\pm\Big(U_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n})\Big) \neq 0$ in order for the action to remain invariant under $Q_\pm$. One can check explicitly that this breaking of supersymmetry occurs in the action.

Related to this is the incompatibility of the supersymmetry transformations $Q_\pm U_\mu(\mathbf{n}) = i\psi_\mu(\mathbf{n})$ with invariances of the functional measure. In the continuum, topological field theories are based on the largest invariance possible,

$$QA_\mu(x) \;=\; \psi_\mu(x)\,, \tag{2.23}$$

of the gauge potential $A_\mu(x)$. This corresponds to the most general shift symmetry of the measure in that case. For the unitary lattice variable $U_\mu(x)$, which should be integrated over the left and right invariant Haar measure, there is no corresponding shift symmetry. Instead, the analog of general shift symmetry corresponds to the most general motion on the unitary group manifold. This is not generated by an ordinary derivative, but by the Lie derivative ∇^a. Infinitesimally, this requires a supersymmetry transformation rule for $U_\mu(\mathbf{n})$ of, for a left derivative,

$$QU_\mu(\mathbf{n}) = i\psi_\mu(\mathbf{n})U_\mu(\mathbf{n})\,, \tag{2.24}$$

and this is indeed the direct lattice analog of the continuum transformation (2.23). The Haar measure is invariant under such a transformation, and it is of course also by construction compatible with the unitarity constraint $U_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}) = 1$. The Haar measure is *not* invariant under the naive rule $QU_\mu = i\psi_\mu$, with U_μ unitary. Supersymmetry is therefore broken in both the action and the measure (and the combination of the two).

Remarkably, lattice Monte Carlo simulations [23] indicate that the actual breaking of supersymmetry with this kind of reduction to the real line, even at quite strong coupling, is almost undetectable. Perhaps the reason is that the degrees of freedom are correctly specified in terms of the "natural" fermionic variables (site variables, link variables, and corner variables), and that the number of bosonic and fermionic degrees match. This issue deserves more attention, as it may point towards new and approximate manners of simulating supersymmetric field theories on the lattice.

3. Comment on a relation to Sugino's lattice action

Very recently, Takimi [24] has shown how a small deformation of Sugino's lattice formulation of two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory [9, 10] can be obtained by a consistent truncation of some of the degrees of freedom in Catterall's model, while still preserving a BRST symmetry. In this section, we make some comments on this truncation. In particular, since we have now realized that there are in fact two scalar supersymmetries, we wish to see what happens if we instead perform a similar truncation that preserves the other (anti-)BRST charge.

Let us first briefly review the idea of ref. [24]. First of all, we regard $U_\mu(\mathbf{n})$ as unitary matrices so that $U_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}) = 1$. By this truncation, we impose hermiticity of $A_\mu(\mathbf{n})$. In order that this truncation is consistent with the BRST transformation by Q_+, we impose

$$Q_+(U_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n})) = 0, \tag{3.1}$$

which leads to

$$\psi_\mu^\dagger(\mathbf{n}) = U_\mu^\dagger(\mathbf{n})\psi_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}), \tag{3.2}$$

or equivalently,

$$\big(\psi_{(\mu)}(\mathbf{n})\big)^\dagger = \psi_{(\mu)}, \qquad \psi_{(\mu)}(\mathbf{n}) \equiv \psi_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}), \tag{3.3}$$

that is, $\psi_{(\mu)}(\mathbf{n})$ are hermitian. Here, the link variables $\psi_\mu(\mathbf{n})$ have been transformed into *site variables* $\psi_{(\mu)}(\mathbf{n})$. Similarly, we define a site variable,

$$\chi(\mathbf{n}) \equiv \chi_{12}(\mathbf{n})U_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n}), \tag{3.4}$$

and impose it to be hermitian. Then, $\chi_{12}^\dagger(\mathbf{n})$ is related to $\chi_{12}(\mathbf{n})$ as

$$\chi_{12}^\dagger(\mathbf{n}) = U_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n})\chi_{12}(\mathbf{n})U_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n}). \tag{3.5}$$

Furthermore, we define a hermitian field $H(\mathbf{n})$ through the relation,

$$B_{12}(\mathbf{n}) = H(\mathbf{n})U_1(\mathbf{n})U_2(\mathbf{n}+\mathbf{e}_1) - i\chi(\mathbf{n})\Big(\psi_1(\mathbf{n})U_2(\mathbf{n}+\mathbf{e}_1) + U_1(\mathbf{n})\psi_2(\mathbf{n}+\mathbf{e}_1)\Big). \tag{3.6}$$

As same as the case of $\chi_{12}^\dagger$, $B_{12}^\dagger$ is determined uniquely by imposing $H(\mathbf{n})$ to be hermitian:

$$B_{12}^\dagger(\mathbf{n}) = U_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n})H(\mathbf{n}) - i\Big(U_2^\dagger(\mathbf{n}+\mathbf{e}_1)\psi_1^\dagger(\mathbf{n}) + \psi_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n})\Big)\chi(\mathbf{n}). \tag{3.7}$$

Finally, we set

$$\eta_+(\mathbf{n}) \equiv \eta(\mathbf{n}), \quad \eta_-(\mathbf{n}) \equiv 0, \quad d(\mathbf{n}) \equiv 0. \tag{3.8}$$

As a result, the BRST transformation (2.15) turns out to be

$$\begin{aligned}
QU_\mu(\mathbf{n}) &= i\psi_{(\mu)}(\mathbf{n})U_\mu(\mathbf{n}),\\
Q\psi_{(\mu)}(\mathbf{n}) &= i\psi_{(\mu)}(\mathbf{n})\psi_{(\mu)}(\mathbf{n}) + \frac{1}{2}\Big(U_\mu(\mathbf{n})\phi(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n}) - \phi(\mathbf{n})\Big),\\
Q\overline{\phi}(\mathbf{n}) &= \eta(\mathbf{n}),\\
Q\eta(\mathbf{n}) &= -\frac{i}{2}[\phi(\mathbf{n}),\overline{\phi}(\mathbf{n})],\\
Q\chi(\mathbf{n}) &= H(\mathbf{n}),\\
QH(\mathbf{n}) &= -\frac{1}{2}[\phi(\mathbf{n}),\chi(\mathbf{n})],
\end{aligned} \tag{3.9}$$

with $Q \equiv Q_+$. This is nothing but the BRST transformation of Sugino's lattice formulation of the two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory [10] and the consistent BRST transformation (2.24) has been automatically derived. One can also show that the action of Catterall's model (2.14) turns out to be almost that of Sugino's model by this truncation of degrees of freedom (for details, see [24]).

An immediate question is whether the anti-BRST symmetry Q_- is preserved or not. One can easily see that this is not the case. In fact, under the truncation adopted above, the anti-BRST transformation of $U_\mu(\mathbf{n})U_\mu(\mathbf{n})^\dagger$ (the combination that equals unity if U_μ is restricted to be unitary) under the action of Q_- is not zero:

$$Q_-(U_\mu(\mathbf{n})U_\mu(\mathbf{n})^\dagger) = 2i\psi_{(\mu)}(\mathbf{n}) \neq 0\ . \tag{3.10}$$

Similarly, we can show that the action of Q_- is incompatible with hermiticity of $\chi(\mathbf{n})$ and $H(\mathbf{n})$ and the conditions $d(\mathbf{n}) = \eta_-(\mathbf{n}) = 0$. Therefore, Q_- is not consistent with the rule of truncation introduced above, and the truncated theory possesses only one preserved BRST charge.

In the above argument, we truncated the degrees of freedom with preserving the BRST symmetry Q_+. However, in principle, we can choose any linear combination of Q_+ and Q_- to be preserved. As example, let us choose Q_- to be preserved. In this case, the relation corresponding to (3.2) is

$$\psi_\mu^\dagger(\mathbf{n}) = -U_\mu^\dagger(\mathbf{n})\psi_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}), \tag{3.11}$$

then we can define hermitian site fermions as,

$$\psi_{(\mu)}(\mathbf{n}) = i\psi_\mu(\mathbf{n})U_\mu^\dagger(\mathbf{n}). \tag{3.12}$$

Similarly, we can define hermitian site variables $\chi(\mathbf{n})$ and $H(\mathbf{n})$ by

$$\chi(\mathbf{n}) = \chi_{12}(\mathbf{n})U_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n}), \tag{3.13}$$

$$H(\mathbf{n}) = iB_{12}(\mathbf{n})U_2^\dagger(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n}) - i\chi(\mathbf{n})\psi_{(2)}(\mathbf{n}+\mathbf{e}_1)U_1^\dagger(\mathbf{n}) - i\chi(\mathbf{n})\psi_{(1)}(\mathbf{n}), \tag{3.14}$$

which is consistent with the BRST transformation (2.21). We can also restrict $\eta_\pm(\mathbf{n})$ and $d(n)$ as

$$\eta_+(\mathbf{n}) \equiv 0, \quad \eta_-(\mathbf{n}) \equiv i\eta(\mathbf{n}), \quad d(\mathbf{n}) \equiv 0. \tag{3.15}$$

By this truncation, we obtain the same BRST transformation as (3.9) after setting $Q \equiv -iQ_-$ and we again obtain the action of Sugino's formulation (plus the additional terms). In this case, the BRST symmetry Q_+ is broken after the truncation. The argument is completely parallel for any linear combination of Q_+ and Q_-,

$$\tilde{Q} \equiv \alpha Q_+ + \beta Q_- \,. \tag{3.16}$$

If $\beta = \pm\alpha$ it seems impossible to impose the condition $U_\mu^\dagger(\mathbf{n})U_\mu(\mathbf{n}) = 1$.

4. Application to four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory

As mentioned in the introduction, we can apply the prescription discussed in the section 2 to any other supersymmetric gauge theory. In particular, it seems to be also applicable to theories that are not described in terms of Dirac-Kähler fermions. In this section, we apply it to four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory as an example.

The starting point of the discussion is the mother theory, that is, the dimensionally reduced theory of the four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills Lagrangian. The purpose is to construct a lattice formulation that possesses at least one supercharge. To this end, we start with the dimensionally reduced action of the topologically twisted four-dimensional $\mathcal{N} = 2$ SYM theory [21]:

$$S = \frac{1}{g^2}\mathrm{Tr}\, Q\left\{-\chi_{\mu\nu}^+\left(B_{\mu\nu}^+ - F_{\mu\nu}\right) - \frac{i}{2}\psi_\mu[A_\mu,\overline{\phi}] + \frac{i}{8}\eta[\phi,\overline{\phi}]\right\}, \tag{4.1}$$

	A_μ	$A_\mu^\dagger$	ϕ	$\overline{\phi}$	$B_{\mu\nu}$	$B_{\mu\nu}^\dagger$	η	ψ_μ	$\psi_\mu^\dagger$	$\chi_{\mu\nu}$	$\chi_{\mu\nu}^\dagger$
$\mathbf{q}$	$\mathbf{e}_\mu$	$-\mathbf{e}_\mu$	0	0	$\mathbf{e}_\mu+\mathbf{e}_\nu$	$-\mathbf{e}_\mu-\mathbf{e}_\nu$	0	$\mathbf{e}_\mu$	$-\mathbf{e}_\mu$	$\mathbf{e}_\mu+\mathbf{e}_\nu$	$-\mathbf{e}_\mu-\mathbf{e}_\nu$

Table 2: The charge assignment for the complexified fields.

where $\mu,\nu = 1,\ldots,4$ and $F_{\mu\nu} \equiv i[A_\mu, A_\nu]$. We have assumed that $\{A_\mu, \overline{\phi}, B^+_{\mu\nu}, \phi\}$ and $\{\psi_\mu, \eta, \chi^+_{\mu\nu}\}$ are bosonic and fermionic hermitian matrices of size kN^4, and $\chi^+_{\mu\nu}$ and $B^+_{\mu\nu}$ are anti-symmetric with respect to the Lorentz indices and satisfy the self-dual condition, $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\chi^+_{\rho\sigma} = \chi^+_{\mu\nu}$ and the same equation for $B^+_{\mu\nu}$. The BRST charge Q acts on the fields as

$$\begin{aligned} QA_\mu &= \psi_\mu, \qquad Q\psi_\mu = -i[A_\mu,\phi], \\ Q\overline{\phi} &= \eta, \qquad Q\eta = i[\phi,\overline{\phi}], \\ Q\chi^+_{\mu\nu} &= B^+_{\mu\nu}, \quad QB^+_{\mu\nu} = i[\phi,\chi^+_{\mu\nu}], \qquad Q\phi = 0. \end{aligned} \tag{4.2}$$

As we did in section 2, we next extend the theory by complexifying the fields A_μ, ψ_μ, $\chi^+_{\mu\nu}$ and $B^+_{\mu\nu}$ in order that the theory has enough U(1) symmetries to create a four-dimensional space-time by orbifolding. In this case, however, the complexification is not sufficient, since the self-duality of the fields $\chi^+_{\mu\nu}$ and $B^+_{\mu\nu}$ makes it impossible to define U(1) charges that are compatible with the first term of the action (4.1). To overcome this problem, we further extend $\chi^+_{\mu\nu}$ and $B^+_{\mu\nu}$ to complex rank 2 tensors without self-dual constraint, $\chi_{\mu\nu}$ and $B_{\mu\nu}$, respectively. After these extension, we obtain the action of "complexified" mother theory:

$$S = \frac{1}{2g^2}\mathrm{Tr}\, Q\left\{-\chi^\dagger_{\mu\nu}\left(B_{\mu\nu}-F_{\mu\nu}\right)-\chi_{\mu\nu}\left(B^\dagger_{\mu\nu}-F^\dagger_{\mu\nu}\right)-\frac{i}{2}\psi^\dagger_\mu[A_\mu,\overline{\phi}]-\frac{i}{2}\psi_\mu[A^\dagger_\mu,\overline{\phi}]+\frac{i}{4}\eta[\phi,\overline{\phi}]\right\}. \tag{4.3}$$

For the fields in this complexified theory, we can assign non-trivial U(1) charges as in table 2, where $\mathbf{q} \equiv (q_1, q_2, q_3, q_4)$ is a set of four U(1) charges and we have defined

$$\mathbf{e}_1 \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \mathbf{e}_2 \equiv \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \mathbf{e}_3 \equiv \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \mathbf{e}_4 \equiv \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{4.4}$$

Correspondingly, we can make an orbifolding by substituting the expansion like (2.11) into the complexified action (4.3). The lattice action is obtained by carrying out the replacement like (2.13) followed by some consistent truncation of the degrees of freedom.

In order to simplify the description, however, we change the order of the prescription in this section; (1) we first replace A_μ to $-iU_\mu$ (deconstruction), (2) we next truncate some degrees of freedom of the complexified matrix theory, and (3) we finally will perform the orbifolding. We can explicitly show that it is equivalent to the prescription discussed in the section 2.

Following this procedure, we first replace A_μ and $A^\dagger_\mu$ by

$$A_\mu \to -iU_\mu, \quad A^\dagger_\mu \to iU^\dagger_\mu. \tag{4.5}$$

Then the action (4.3) becomes

$$S=\frac{1}{2g^2}\mathrm{Tr}\,Q\Big\{-\chi^\dagger_{\mu\nu}\Big(B_{\mu\nu}-\mathcal{F}_{\mu\nu}\Big)-\chi_{\mu\nu}\Big(B^\dagger_{\mu\nu}-\mathcal{F}^\dagger_{\mu\nu}\Big)-\frac{1}{2}\psi^\dagger_\mu[U_\mu,\overline{\phi}]+\frac{1}{2}\psi_\mu[U^\dagger_\mu,\overline{\phi}]+\frac{i}{4}\eta[\phi,\overline{\phi}]\Big\}, \tag{4.6}$$

where $\mathcal{F}_{\mu\nu}$ is given by

$$\mathcal{F}_{\mu\nu}=-i[U_\mu,U_\nu], \tag{4.7}$$

and the BRST transformation (4.2) becomes

$$\begin{aligned} QU_\mu&=i\psi_\mu, & Q\psi_\mu&=-[U_\mu,\phi],\\ QU^\dagger_\mu&=-i\psi^\dagger_\mu, & Q\psi^\dagger_\mu&=[U^\dagger_\mu,\phi],\\ Q\overline{\phi}&=\eta, & Q\eta&=i[\phi,\overline{\phi}],\\ Q\chi_{\mu\nu}&=B_{\mu\nu}, & QB_{\mu\nu}&=i[\phi,\chi_{\mu\nu}],\\ Q\chi^\dagger_{\mu\nu}&=B^\dagger_{\mu\nu}, & QB^\dagger_{\mu\nu}&=i[\phi,\chi^\dagger_{\mu\nu}], & Q\phi&=0. \end{aligned} \tag{4.8}$$

Next, we must truncate some degrees of freedom. As discussed in the section 2.3, the naive restriction to "real line" breaks not only the remaining supersymmetry but also the gauge symmetry of the system, in general. Thus, it seems to be better to adopt the way of truncation adopted in [24]. We first impose U_μ to be unitary matrices. Then, repeating the same discussion around (3.2), we can show that $\psi^\dagger_\mu$ is related to ψ_μ as

$$\psi^\dagger_\mu=U^\dagger_\mu\psi_\mu U^\dagger_\mu, \tag{4.9}$$

and we can define hermitian matrices $\psi_{(\mu)}$ as

$$\psi_{(\mu)}\equiv\psi_\mu U^\dagger_\mu. \tag{4.10}$$

In (4.9) and (4.10), we do *not* sum over μ. In the following, we do not sum over duplicated symbols unless we explicitly write it.

In order to truncate the half of the degrees of freedom of $\chi_{\mu\nu}$ we define complex fermionic fields $\chi_{(\mu\nu)}$ with zero U(1) charges as

$$\chi_{(\mu\nu)}=\begin{cases}\chi_{\mu\nu}U^\dagger_\nu U^\dagger_\mu, & \text{for }\ (\mu,\nu)\in\mathcal{I}\\ -\chi_{\nu\mu}U^\dagger_\mu U^\dagger_\nu, & \text{for }\ (\mu,\nu)\notin\mathcal{I}\end{cases} \tag{4.11}$$

where

$$\mathcal{I}\equiv\{(1,4),\ (2,4),\ (3,4),\ (2,3),\ (3,1),\ (1,2)\}, \tag{4.12}$$

and impose $\chi_{(\mu\nu)}$ to be hermitian. The new field $\chi_{(\mu\nu)}$ satisfies $\chi_{(\mu\nu)}=-\chi_{(\nu\mu)}$ by definition. Note that we can impose the hermiticity only for those fields which have zero U(1) charges. Correspondingly, we define bosonic hermitian anti-symmetric tensor field $H_{(\mu\nu)}$ through the BRST transformation:

$$Q\chi_{(\mu\nu)}\equiv H_{(\mu\nu)}. \tag{4.13}$$

The original fields $\chi_{\mu\nu}$ and $B_{\mu\nu}$ can be expressed by the new fields as

$$\begin{aligned} \psi_\mu &= \psi_{(\mu)} U_\mu, \\ \chi_{\mu\nu} &= \chi_{(\mu\nu)} U_\mu U_\nu, \qquad (\text{for } (\mu,\nu) \in \mathcal{I}) \\ B_{\mu\nu} &= H_{(\mu\nu)} U_\mu U_\nu - i\chi_{(\mu\nu)} \Big(U_\mu \psi_{(\mu)} U_\nu + \psi_{(\mu)} U_\mu U_\nu \Big). \end{aligned} \tag{4.14}$$

We must further restrict the degrees of freedom of $\chi_{(\mu\nu)}$ and $H_{(\mu\nu)}$, and it seems to be proper to impose the self-dual condition to them:

$$\frac{1}{2}\sum_{\rho,\sigma=1}^{4} \epsilon_{\mu\nu\rho\sigma}\chi_{(\rho\sigma)} = \chi_{(\mu\nu)}, \quad \frac{1}{2}\sum_{\rho,\sigma=1}^{4} \epsilon_{\mu\nu\rho\sigma}H_{(\rho\sigma)} = H_{(\mu\nu)}. \tag{4.15}$$

From now on, we denote the three independent components of $\chi_{(\mu\nu)}$ and $H_{(\mu\nu)}$ as

$$\begin{aligned} \vec{\chi} &\equiv (\chi_1, \chi_2, \chi_3) \equiv \left(2\chi_{(14)}, 2\chi_{(24)}, 2\chi_{(34)}\right), \\ \vec{H} &\equiv (H_1, H_2, H_3) \equiv \left(2H_{(14)}, 2H_{(24)}, 2H_{(34)}\right), \end{aligned} \tag{4.16}$$

After the above truncation, the action (4.6) becomes

$$S = \frac{1}{g^2}\mathrm{Tr}\, Q\bigg\{ -\vec{\chi}\cdot\left(\vec{H}+\vec{\Phi}\right) + \frac{1}{2}\sum_{\mu=1}^{4}\psi_{(\mu)}\left(\overline{\phi} - U_\mu\overline{\phi}U_\mu^\dagger\right) + \frac{i}{8}\eta[\phi,\overline{\phi}] + \frac{i}{2}\sum_{i=1}^{3}\chi_i\Psi_i\chi_i \bigg\}, \tag{4.17}$$

where $\vec{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ is given by

$$\begin{aligned} \Phi_1 &= \frac{i}{2}\left(U_{14} - U_{41} + U_{23} - U_{32}\right), \\ \Phi_2 &= \frac{i}{2}\left(U_{24} - U_{42} + U_{31} - U_{13}\right), \\ \Phi_3 &= \frac{i}{2}\left(U_{34} - U_{43} + U_{12} - U_{21}\right), \end{aligned} \tag{4.18}$$

with

$$U_{\mu\nu} \equiv U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger, \tag{4.19}$$

and $\vec{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ is given by

$$\begin{aligned} \Psi_1 &= \mathcal{L}_4^+\psi_{(1)} + \mathcal{L}_1^+\psi_{(4)} + \mathcal{L}_3^+\psi_{(2)} + \mathcal{L}_2^+\psi_{(3)}, \\ \Psi_2 &= \mathcal{L}_4^+\psi_{(2)} + \mathcal{L}_2^+\psi_{(4)} + \mathcal{L}_1^+\psi_{(3)} + \mathcal{L}_3^+\psi_{(1)}, \\ \Psi_3 &= \mathcal{L}_4^+\psi_{(3)} + \mathcal{L}_3^+\psi_{(4)} + \mathcal{L}_2^+\psi_{(1)} + \mathcal{L}_1^+\psi_{(2)}, \end{aligned} \tag{4.20}$$

where

$$\mathcal{L}_\nu^+\psi_{(\mu)} \equiv \psi_{(\mu)} + U_\nu\psi_{(\mu)}U_\nu^\dagger. \tag{4.21}$$

The BRST transformation (4.8) becomes

$$\begin{aligned} &QU_\mu = i\psi_{(\mu)}U_\mu, && Q\psi_{(\mu)} = \phi - U_\mu\phi U_\mu^\dagger + i\psi_{(\mu)}\psi_{(\mu)}, \\ &Q\overline{\phi} = \eta, && Q\eta = i[\phi,\overline{\phi}], \\ &Q\vec{\chi} = \vec{H}, && Q\vec{H} = i[\phi,\vec{\chi}], \qquad Q\phi = 0. \end{aligned} \tag{4.22}$$

Finally, we generate a lattice action from the truncated action (4.17) by orbifolding. By construction, the U(1) charges of U_μ are given by $\mathbf{e}_\mu$ and those of other fields are zero. Then, the orbifold projection is achieved by substituting the following expansions into the truncated action (4.17):

$$
\begin{aligned}
U_\mu &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} U_\mu(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}+\mathbf{e}_\mu}, & U_\mu^\dagger &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} U_\mu^\dagger(\mathbf{n})\otimes E_{\mathbf{n}+\mathbf{e}_\mu,\mathbf{n}},\\
\phi &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} \phi(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}}, & \overline{\phi} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} \overline{\phi}(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}},\\
\psi_{(\mu)} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} \psi_{(\mu)}(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}}, & \eta &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} \eta(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}},\\
\vec{\chi} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} \vec{\chi}(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}}, & \vec{H} &= \sum_{\mathbf{n}\in\mathbb{Z}_N^4} \vec{H}(\mathbf{n})\otimes E_{\mathbf{n},\mathbf{n}},
\end{aligned}
\tag{4.23}
$$

where link variables $U_\mu(\mathbf{n})$ take values in $\mathrm{U}(k)$ and the other lattice fields are hermitian matrices with the size k. As a result, we obtain the action of a lattice formulation for the topologically twisted four-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory:

$$
\begin{aligned}
S = \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{n}\in\mathbb{Z}_N^4} Q\Big\{&-\vec{\chi}(\mathbf{n})\cdot\Big(\vec{H}(\mathbf{n})+\vec{\Phi}(\mathbf{n})\Big)\\
&+\frac{1}{2}\sum_{\mu=1}^{4}\psi_{(\mu)}(\mathbf{n})\Big(\overline{\phi}(\mathbf{n})-U_\mu(\mathbf{n})\overline{\phi}(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n})\Big)\\
&+\frac{i}{8}\eta(\mathbf{n})[\phi(\mathbf{n}),\overline{\phi}(\mathbf{n})]+\frac{i}{2}\sum_{i=1}^{3}\chi_i(\mathbf{n})\Psi_i(\mathbf{n})\chi_i(\mathbf{n})\Big\},
\end{aligned}
\tag{4.24}
$$

where

$$
\begin{aligned}
\Phi_1(\mathbf{n}) &= \frac{i}{2}\left(U_{14}(\mathbf{n})-U_{41}(\mathbf{n})+U_{23}(\mathbf{n})-U_{32}(\mathbf{n})\right),\\
\Phi_2(\mathbf{n}) &= \frac{i}{2}\left(U_{24}(\mathbf{n})-U_{42}(\mathbf{n})+U_{31}(\mathbf{n})-U_{13}(\mathbf{n})\right),\\
\Phi_3(\mathbf{n}) &= \frac{i}{2}\left(U_{34}(\mathbf{n})-U_{43}(\mathbf{n})+U_{12}(\mathbf{n})-U_{21}(\mathbf{n})\right),
\end{aligned}
\tag{4.25}
$$

with

$$
U_{\mu\nu}(\mathbf{n}) \equiv U_\mu(\mathbf{n})U_\nu(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n}+\mathbf{e}_\nu)U_\nu^\dagger(\mathbf{n}),
\tag{4.26}
$$

and

$$
\begin{aligned}
\Psi_1 &= \mathcal{L}_4^+\psi_{(1)}(\mathbf{n})+\mathcal{L}_1^+\psi_{(4)}(\mathbf{n})+\mathcal{L}_3^+\psi_{(2)}(\mathbf{n})+\mathcal{L}_2^+\psi_{(3)}(\mathbf{n}),\\
\Psi_2 &= \mathcal{L}_4^+\psi_{(2)}(\mathbf{n})+\mathcal{L}_2^+\psi_{(4)}(\mathbf{n})+\mathcal{L}_1^+\psi_{(3)}(\mathbf{n})+\mathcal{L}_3^+\psi_{(1)}(\mathbf{n}),\\
\Psi_3 &= \mathcal{L}_4^+\psi_{(3)}(\mathbf{n})+\mathcal{L}_3^+\psi_{(4)}(\mathbf{n})+\mathcal{L}_2^+\psi_{(1)}(\mathbf{n})+\mathcal{L}_1^+\psi_{(2)}(\mathbf{n}),
\end{aligned}
\tag{4.27}
$$

with

$$
\mathcal{L}_\nu^+\psi_{(\mu)}(\mathbf{n}) \equiv \psi_{(\mu)}(\mathbf{n})+U_\nu(\mathbf{n})\psi_{(\mu)}(\mathbf{n}+\mathbf{e}_\nu)U_\nu^\dagger(\mathbf{n}).
\tag{4.28}
$$

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The BRST transformation is given by

$$\begin{aligned} &QU_\mu(\mathbf{n}) = i\psi_{(\mu)}(\mathbf{n})U_\mu(\mathbf{n}), \quad Q\psi_{(\mu)}(\mathbf{n}) = \phi(\mathbf{n}) - U_\mu(\mathbf{n})\phi(\mathbf{n}+\mathbf{e}_\mu)U_\mu^\dagger(\mathbf{n}) + i\psi_{(\mu)}(\mathbf{n})\psi_{(\mu)}(\mathbf{n}), \\ &Q\overline{\phi}(\mathbf{n}) = \eta(\mathbf{n}), \qquad Q\eta(\mathbf{n}) = i[\phi(\mathbf{n}), \overline{\phi}(\mathbf{n})], \\ &Q\vec{\chi}(\mathbf{n}) = \vec{H}(\mathbf{n}), \qquad Q\vec{H}(\mathbf{n}) = i[\phi(\mathbf{n}), \vec{\chi}(\mathbf{n})], \qquad Q\phi(\mathbf{n}) = 0. \end{aligned} \tag{4.29}$$

Again, the obtained lattice action (4.24) is almost that of Sugino's formulation for four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory given in [9], and the only difference is the existence of the last terms of (4.24). Thus, we conclude that, as in the case of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory, Sugino's lattice formulation of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory can also be derived from the dimensionally reduced matrix model by using the orbifolding prescription together with a proper sequence of extension and truncation of the degrees of freedom.

5. Conclusion

In this paper, we have shown that Catterall's lattice formulations can be understood in terms of the orbifolding procedure. We have explicitly demonstrated this by a derivation of Catterall's model based on a complexified matrix model as a mother theory. The symmetry of the mother theory is enhanced by this complexification, and Catterall's model possesses in fact two independent BRST symmetries. We have also commented on the relationship between Catterall's model and a variant of Sugino's lattice formulation of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory as derived in [24]. We have shown that we can restrict the degrees of freedom of Catterall's model so that a linear combination of the two BRST charges, $\alpha Q_+ + \beta Q_-$ ($\beta \neq \pm\alpha$), is preserved. The restricted theory does not depend on the values of α and β after trivial redefinitions. We have also applied the procedure developed in section 2 to topologically twisted four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. The lattice theory obtained in that manner is related to Sugino's formulation [9] up to the same kind of terms that were found in the two-dimensional case.

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